DISSIPATIVE DISTRIBUTED SYSTEMS

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OUTLINE

- **Dissipative dynamical systems**
- **The construction of storage functions**
- **A physical example**
- **Distributed differential systems**
- **Controllability**
- **Quadratic differential forms**
- **Global versus local dissipation laws**

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LYAPUNOV FUNCTIONS

Consider the classical 'dynamical system', the flow

$$\frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X}$, the state space. The function

$$V : \mathbb{X} \to \mathbb{R}$$

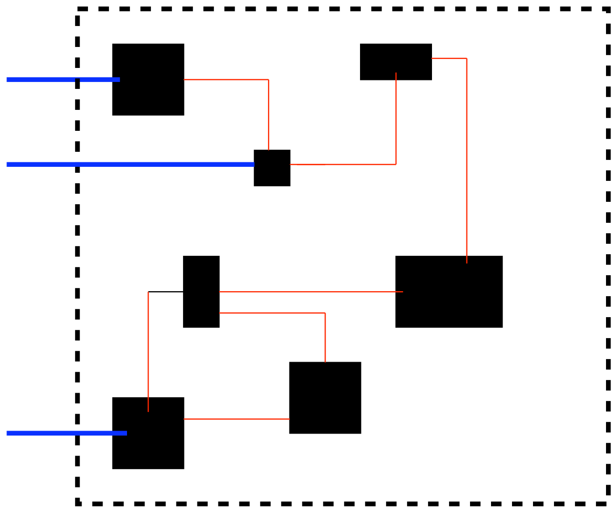
is said to be a Lyapunov function if along trajectories

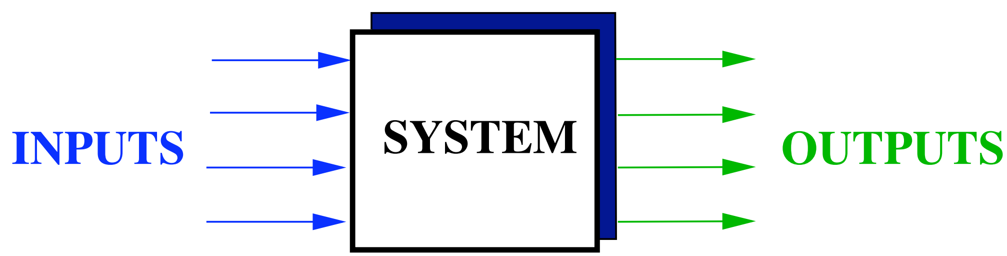
$$\frac{d}{dt}V(x) \leq 0$$

Equivalent to $\nabla V(x) \cdot f(x) \leq 0$ **for all** $x \in \mathbb{X}$.

Very useful idea in stability analysis.

A much more appropriate starting point for the study of dynamics are open systems. $\rightsquigarrow$





INPUT/STATE/OUTPUT SYSTEMS

Consider

$$\Sigma : \quad \frac{d}{dt}x = f(x,u); \quad y = h(x,u).$$

$u \in \mathbb{U}$, $x \in \mathbb{X}$, $y \in \mathbb{Y}$: **the input, state, and output.**

Let

$$s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$$

be a function, called the ***supply rate.***

$s(u,y)$ **models something like the power delivered to the system when the input value is** u **and output value is** y**.**

DISSIPATIVITY

Σ is defined to be *dissipative* w.r.t. the supply rate s if there exists

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function*, such that

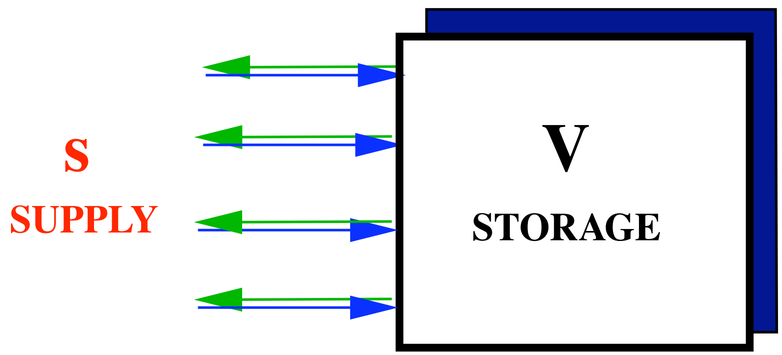
$$\frac{d}{dt} V(x) \leq s(u, y)$$

along input/state/output trajectories of Σ.

This inequality is called the *dissipation inequality.*

Equivalent to $\nabla V(x) \cdot f(x, u) \leq s(u, h(x, u))$ **for all** $(u, x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: 'conservative' system.



Dissipativeness $\Leftrightarrow$ Increase in storage $\leq$ Supply.

EXAMPLES

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v$ F : force, v : velocity	potential + kinetic energy
Thermodynamic system	$Q + W$ Q : heat, W : work	internal energy
Thermodynamic system	$-Q/T$ Q : heat, T : temp.	entropy
etc.	etc.	etc.

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- **Dissipative dynamical systems**
- **The construction of storage functions: LMI's**
- **A physical example**
- **Distributed differential systems**
- **Controllability**
- **Quadratic differential forms**
- **Global versus local dissipation laws**

THE CONSTRUCTION OF STORAGE FUNCTIONS

Central question: Given (a representation of) Σ and s,

does there exist V such that the dissipation inequality holds?

Assume Σ linear, time-invariant, finite-dimensional, minimal:

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx;$$

and s quadratic: e.g.,

$$s : (u, y) \mapsto |u|^2 - |y|^2.$$

Then V exists iff $\forall\ u \in \mathfrak{L}_2, \exists\ y \in \mathfrak{L}_2$ (unique) such that

$$||y||_{\mathfrak{L}_2} \leq ||u||_{\mathfrak{L}_2}.$$

... iff a quadratic one exists: $V(x) = x^\top K x$.

... iff there exists a solution $K = K^\top$ to the

Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^\top K + KA - C^\top C & KB \\ B^\top K & -I \end{bmatrix} \leq 0.$$

.. iff ... to the Algebraic Riccati Inequality

$$A^\top K + KA - KBB^\top K - C^\top C \leq 0.$$

... to the Algebraic Riccati Equation (ARE)

$$A^\top K + KA - KBB^\top K - C^\top C = 0.$$

Solution set is convex, compact, and attains its inf and sup:

$$K^- \leq K \leq K^+$$

Extensive theory, relation with other system representations, many applications, well-understood (also algorithmically).

The notion of a dissipative system:

- **Generalization of 'Lyapunov function' to open systems**
- **Central concept in control theory: many applications to feedback stability, robust ($\mathcal{H}_\infty$-) control, adaptive control, system identification, passivation control**
- **$\rightsquigarrow$ passive electrical circuit synthesis procedures**
- **Notable special case: second law of thermodynamics**
- **Drawback 1: requires separation of interaction variables in inputs and outputs**
- **Drawback 2: imposes storage function = state function.**
  This is something one would like to prove!
- **Drawback 3: limited to dynamical ($\leftrightarrow$ distributed) systems**

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- **A physical example: heat diffusion**
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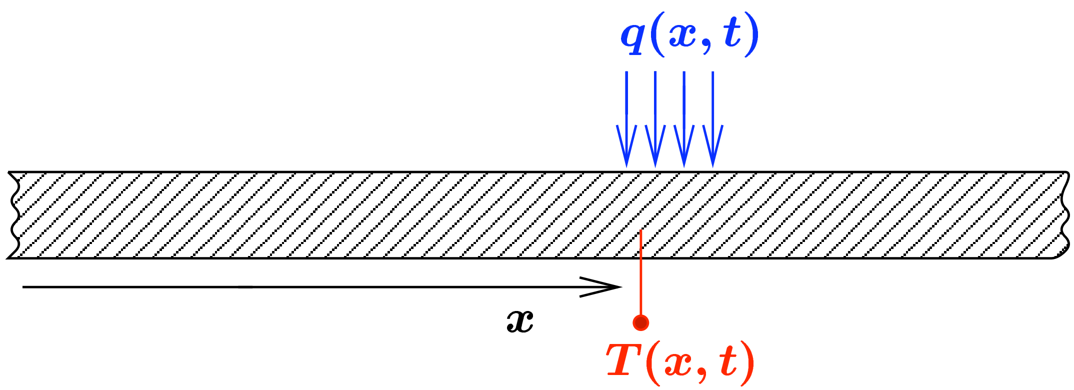
HEAT DIFFUSION



Evolution of the temperature along a heat conducting bar:

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q, \quad T > 0,$$

$T(x,t) \in \mathbb{R}$ **: temperature,**

$q(x,t) \in \mathbb{R}$ **: rate of heat supplied.**

First law:

Diffusion is conservative w.r.t. the supply rate q:

$$\boxed{\int_{\mathbb{R}^2} q(t,x)\, dx\, dt \; = \; 0,}$$

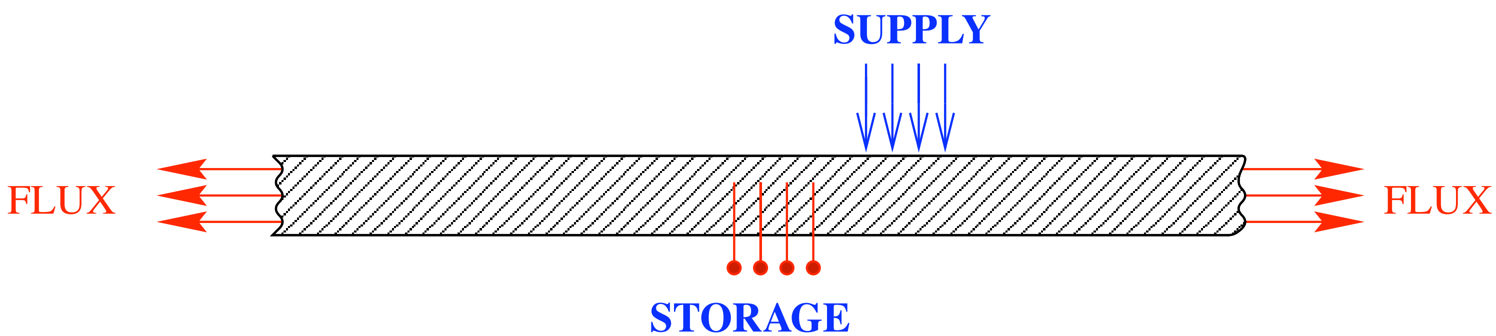
Second law:

Diffusion is dissipative w.r.t. the supply rate $-\frac{q}{T}$.

$$\boxed{\int_{\mathbb{R}^2} \frac{q(t,x)}{T(x,t)}\, dx\, dt \; \leq \; 0.}$$

for all $(T,q) \in \mathfrak{B}_{T_0}$ = the elements of the behavior such that $T(t,x) = T_0 > 0$ outside a compact set.

Can these 'global' versions be expressed as 'local' laws?



Change of storage + Flux $\leq$ Supply.

Define the following variables:

$$E = T \quad : \quad \text{the stored energy density,}$$

$$F_E = -\frac{\partial}{\partial x} T \quad : \quad \text{the energy flux,}$$

$$S = \ln(T) \quad : \quad \text{the entropy density,}$$

$$F_S = -\frac{1}{T}\frac{\partial}{\partial x} T \quad : \quad \text{the entropy flux,}$$

$$D_S = (\frac{1}{T}\frac{\partial}{\partial x} T)^2 \quad : \quad \text{the rate of entropy production.}$$

Local versions **of the first and second law:**

Conservation of energy:

$$\boxed{\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} F_E = q,}$$

Entropy production:

$$\boxed{\frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S = \frac{q}{T} + D_S.}$$

Note: Since $(D_S \geq 0) \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S \geq \frac{q}{T}.}$

It is this *ad hoc* construction of the

storage and flux

that we want to systematize, to carry out in general.

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BEHAVIORAL SYSTEMS

A system **:=** $\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T}$ $=$ **the set of *independent* variables**

time, space, time and space

$\mathbb{W}$ $=$ **the set of *dependent* variables**

(= where the variables take on their values),

signal space, space of field variables, . . .

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **:** ***the behavior*** **= the admissible trajectories**

$$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

for a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,
$w \notin \mathfrak{B}$: the model forbids the trajectory w.

In this lecture, $\mathbb{T} = \mathbb{R}^n$, ('$\mathtt{n}$-D systems'), $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$$w : \mathbb{R}^n \to \mathbb{R}^{\mathtt{w}}, (w_1(x_1, \cdots, x_{\mathtt{n}}), \cdots, w_{\mathtt{w}}(x_1, \cdots, x_{\mathtt{n}})),$$

often, $\mathtt{n} = 4$, independent variables (t, x, y, z),

$\mathfrak{B}$ = solutions of a system of constant coefficient linear PDE's.

'Linear shift-invariant distributed differential systems'.

Example: *Maxwell's equations*

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space),

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

n-D LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^n, \quad \mathbb{W} = \mathbb{R}^{\mathtt{w}},$

$\mathfrak{B}$ = **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**, and consider**

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)}$$

Define

$$\boxed{\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}}$$

$\mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**

<u>Notation</u>: $\mathtt{n}$-D linear differential systems:

$$(\mathbb{R}^{n}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}},$$

$$\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right).$$

Example:

Maxwell's equations

$$w = \text{col}[E_x, E_y, E_z, B_x, B_y, B_z, j_x, j_y, j_z, \rho],$$

$R \in \mathbb{R}^{8\times 10}[\xi_t, \xi_x, \xi_y, \xi_x]$:

$$R(\xi_t, \xi_x, \xi_y, \xi_z) = \begin{bmatrix} \xi_x & \xi_y & \xi_z & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon_0} \\ 0 & -\xi_z & \xi_y & \xi_t & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_z & 0 & -\xi_x & 0 & \xi_t & 0 & 0 & 0 & 0 & 0 \\ -\xi_y & \xi_x & 0 & 0 & 0 & \xi_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_x & \xi_y & \xi_z & 0 & 0 & 0 & 0 \\ \xi_t & 0 & 0 & 0 & \xi_z & -\xi_y & \frac{1}{\varepsilon_0} & 0 & 0 & 0 \\ 0 & \xi_t & 0 & -\xi_z & 0 & \xi_x & 0 & \frac{1}{\varepsilon_0} & 0 & 0 \\ 0 & 0 & \xi_t & \xi_y & -\xi_x & 0 & 0 & 0 & \frac{1}{\varepsilon_0} & 0 \end{bmatrix},$$

$\mathfrak{B} \in \mathfrak{L}_4^1$.

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CONTROLLABILITY

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **is said to be**

controllable

if for all $w_1, w_2 \in \mathfrak{B}$ **and**

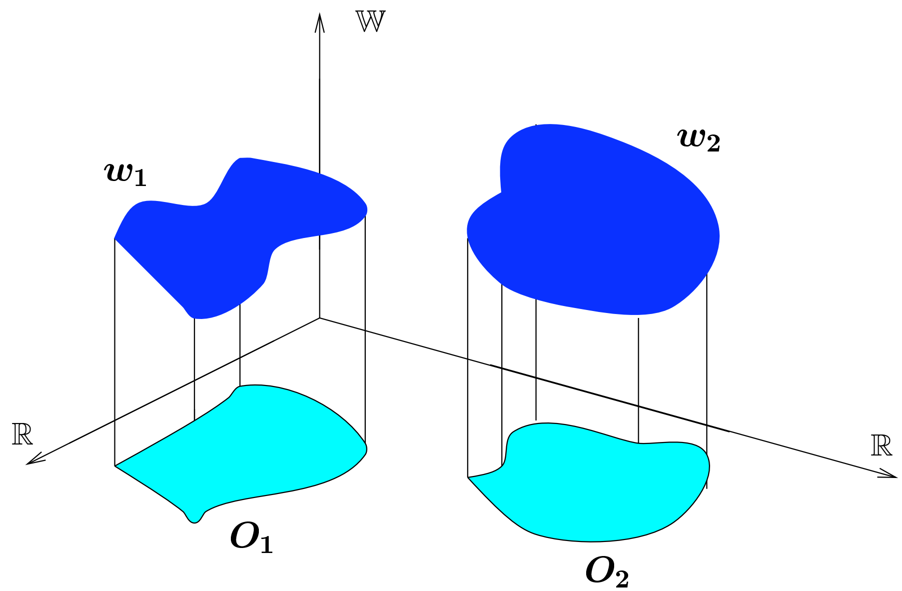
for all $O_1, O_2 \subset \mathbb{R}^n$**, non-overlapping closure,**

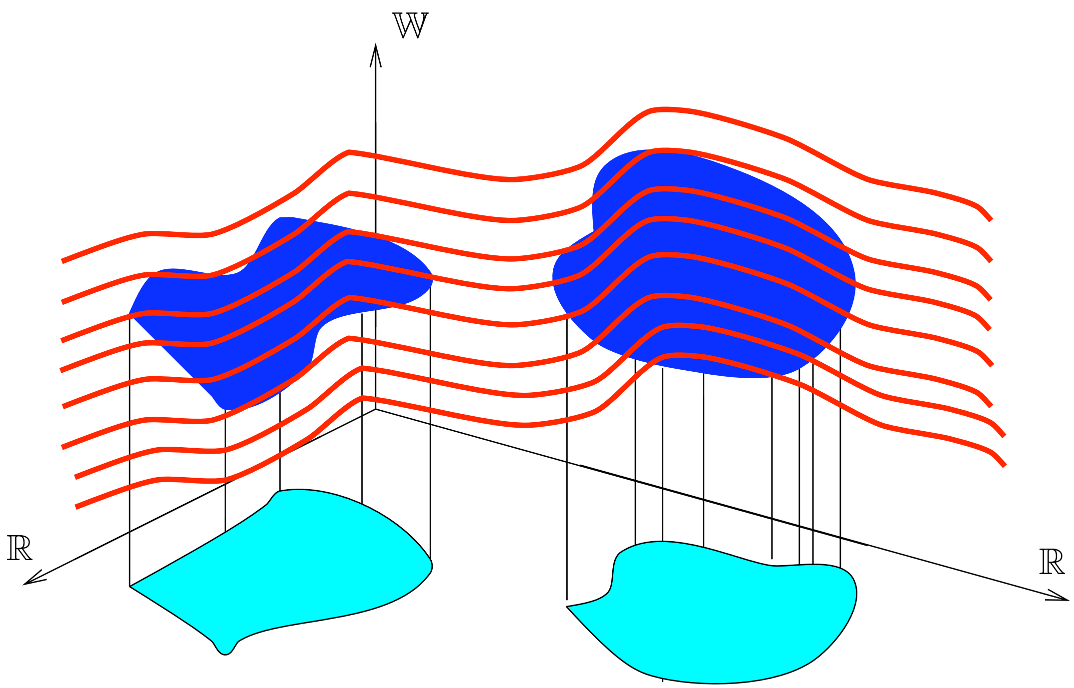
there exists $w \in \mathfrak{B}$ **such that** $w|_{O_1} = w_1|_{O_1}$ **and** $w|_{O_2} = w_2|_{O_2}$**.**

I.e., controllability :$\Leftrightarrow$ **the elements of** $\mathfrak{B}$ **are ‘patch-able’.**

Special case: Kalman controllability for 1-D input/state systems.

In pictures:





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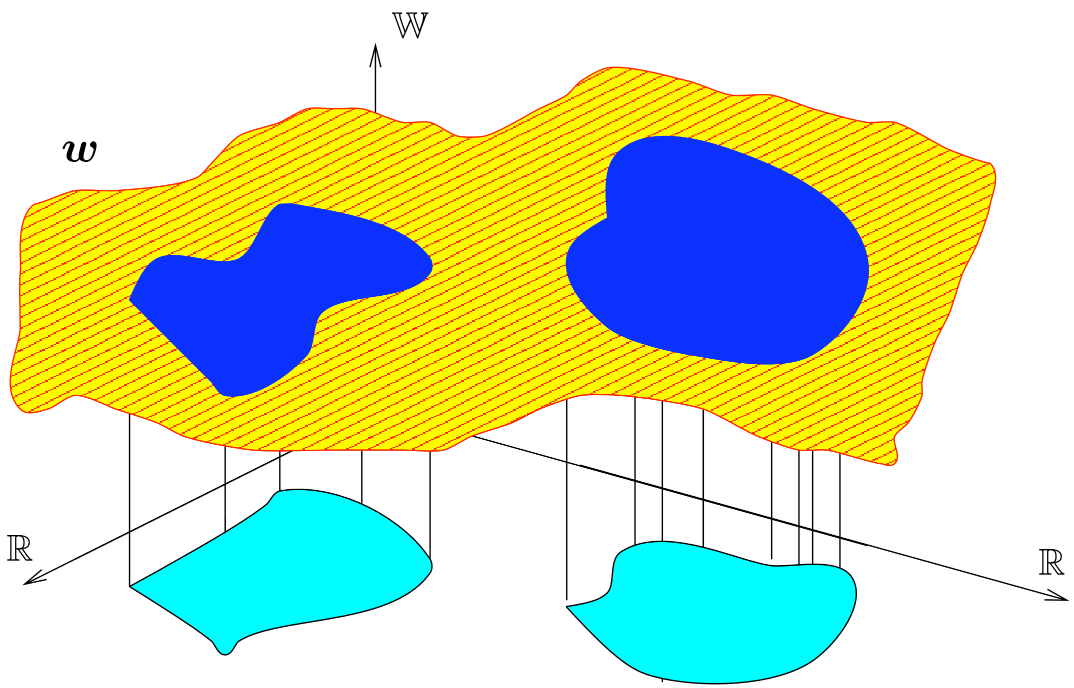
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CONDITIONS FOR CONTROLLABILITY

Representations of elements of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0 \qquad (*)$$

called a *kernel representation* of $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}))$;

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell \quad (***)$$

called an *image representation* of $\mathfrak{B} = \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}))$.

Elimination theorem $\Rightarrow$

every image (of a linear constant coefficient PDO) is also a kernel.

¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

1. **$\mathfrak{B}$ is controllable,**
2. **$\mathfrak{B}$ admits an image representation,**
3. **for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,**
$$a^\top[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals 0 or all of } \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}),$$
4. **$\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,**

etc.

ARE MAXWELL'S EQUATIONS CONTROLLABLE ?

The following equations in the ***scalar potential*** $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ **and the** ***vector potential*** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$**, generate exactly the solutions to Maxwell's equations:**

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

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OBSERVABILITY

Consider

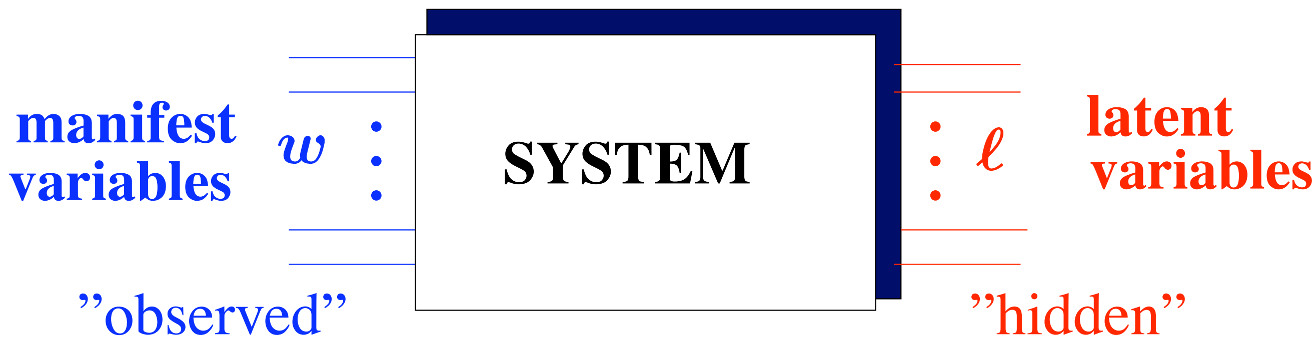
$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell.$$

behavior $\mathfrak{B} = \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}) \in \mathfrak{L}_n^{\mathtt{w}}$,
w : manifest variables ; ℓ : latent variables.

<u>Definition</u>: ℓ is said to be

observable from w

if to each $w \in \mathfrak{B}$, there exists a unique ℓ such that $w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$, i.e., iff $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ is injective.


manifest variables
w
SYSTEM
ℓ
latent variables
"observed"
"hidden"

Controllability $\Rightarrow$ ¿¿ $\exists$ an observable image representation ??

For 1-D systems, yes! For $\mathtt{n}$-D systems, not necessarily!

Call $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ *trivializable* if $\exists$ an invertible PDO,
$U(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})$**, (i.e., with $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ unimodular) such that**

$$U(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = \{(w_1', w_2') \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid w_2' = 0\}.$$

Hence the behavior of $(w_1', w_2') \in U(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B}$ is then:

w_1' **free,** $w_2' = 0$**.**

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$,
$\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$:

1. $\mathfrak{B}$ admits an observable image representation,
2. $\mathfrak{B}$ is trivializable,
3. $\operatorname{rank}(R(\lambda_1, \cdots, \lambda_{\mathtt{n}})) = \operatorname{rank}(R)$
   for all $(\lambda_1, \cdots, \lambda_{\mathtt{n}}) \in \mathbb{C}^n$.

Proof: Serre's conjecture.

Example: controllable 1-D systems.

Non-examples: $\ker(\nabla \times) = \operatorname{im}(\nabla)$, $\ker(\nabla \cdot) = \operatorname{im}(\nabla \times)$.

Non-example: Maxwell's equations. Potential is not observable!
No potential ever is, for Maxwell's equations.

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Multi-index notation:

$x = (x_1, \dots, x_n)$,

$k = (k_1, \dots, k_n), \ell = (\ell_1, \dots, \ell_n)$,

$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \dots, \zeta_n), \eta = (\eta_1, \dots, \eta_n)$,

$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \dots, \frac{\partial^{k_n}}{\partial x_n^{k_n}})$,

$dx = dx_1 dx_2 \dots dx_n$.

QDF's

Let $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ (only finite number $\neq 0$).
The map from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ defined by

$$w \mapsto \sum_{k,\ell} (\frac{d^k}{dx^k} w)^\top \Phi_{k,\ell} (\frac{d^\ell}{dx^\ell} w)$$

is called *quadratic differential form (QDF)* on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ defined by

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote this QDF as Q_Φ; whence $Q_\Phi : w \mapsto Q_\Phi(w)$.
This map is parameterized by $\Phi(\zeta, \eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$.

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DISSIPATIVE DISTRIBUTED SYSTEMS

We only consider controllable linear differential systems and QDF's.

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, **controllable, is said to be** ***dissipative*** **with respect to the** **supply rate** Q_Φ **(a QDF) if**

$$\int_{\mathbb{R}^n} Q_\Phi(w)dx \geq 0$$

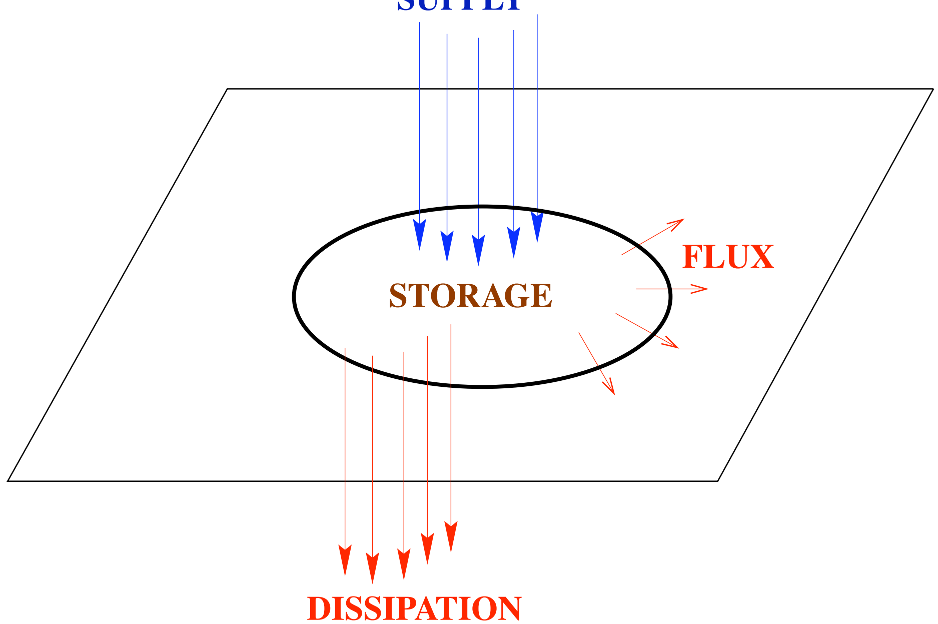
for all $w \in \mathfrak{B}$ **of compact support.**

Idea: $Q_\Phi(w)(x_1, \ldots, x_{\mathtt{n}})dx_1 \cdots dx_{\mathtt{n}}$ **: rate of 'energy' delivered to the system.**

Dissipativity :$\Leftrightarrow$ **the system absorbs (in space and time) net energy.**

LOCAL VERSION

Assume that a system is 'globally' conservative or dissipative.
Can this dissipativity be expressed through a 'local' law?



LOCAL DISSIPATION LAW

Main Theorem: Let $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ be controllable. Then $\mathfrak{B}$ is dissipative with respect to the *supply rate* Q_Φ iff there exist an image representation $w = M(\frac{d}{dx})\ell$ of $\mathfrak{B}$, and an $\mathtt{n}$−vector of QDF's $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_\mathtt{n}})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)})$, called the *flux,* such that the *local dissipation law*

$$\boxed{\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{d}{dx})\ell$.

As usual $\nabla \cdot Q_\Psi := \frac{\partial}{\partial x_1} Q_{\Psi_1} + \cdots + \frac{\partial}{\partial x_\mathtt{n}} Q_{\Psi_\mathtt{n}}$.

Note: the local law involves involves

(possibly unobservable, - i.e., hidden!) latent variables (the ℓ's).

In the case that the independent variables are (t, x, y, z), this theorem can be reformulated as:

$\exists$ ***storage*** $S : \mathbb{R}^{\dim(\ell)} \to \mathbb{R}$, **and**
spatial flux $F : \mathbb{R}^{\dim(\ell)} \to \mathbb{R}^3$, **such that**

$$\boxed{\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$.

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ **the rate of energy supplied.**

Introduce the *stored energy density,* S, and

the *energy flux density* (the *Poynting vector,*) $\vec{F}$, given by

$$S(\vec{E},\vec{B}) = \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) = \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) + \vec{E}\cdot\vec{j} = 0.}$$

Note that the local version of conservation of energy involves $\vec{B}$ in addition to $\vec{E}$ and $\vec{j}$, the variables in the rate of energy supplied.

WHICH PDE's DESCRIBE $(\vec{E}, \vec{j})$ IN MAXWELL'S EQNS ?

Eliminate $\vec{B}$, ρ from Maxwell's equations. Straightforward computation of the relevant left syzygy yields

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Elimination theorem $\Rightarrow$ this exercise would be exact & successful.

IDEA OF THE PROOF

Use controllability and an image representation $w = M'(\frac{d}{dx})\ell'$ to reduce dissipativeness to

$$\int_{\mathbb{R}^n} Q_{\Phi'}(\ell')\, dx \geq 0$$

for all $\ell' \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\bullet)$ of compact support, with

$$\Phi'(\zeta, \eta) = (M')^\top(\zeta)\Phi(\zeta, \eta)M'(\eta).$$

Easily seen to be equivalent to

$$\Phi'(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n.$$

POLYNOMIAL MATRIX FACTORIZATION

Consider the factorization equation

$$\boxed{\Phi'(-\xi,\xi) = X^\top(-\xi)X(\xi).}$$

with $\Phi' \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, $\Phi'(\xi) = (\Phi')^\top(-\xi)$ given, and X the unknown.

For $\mathtt{n} = 1$, it is well-known (but non-trivial) that this factorization equation is solvable (for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$\Phi'(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n.$$

For $\mathtt{n} > 1$ this equation cannot in general be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

This factorizability is a consequence of **Hilbert's 17-th problem!**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, $p \geq 0$, **can in general not** be expressed as

$$p = p_1^2 + p_2^2 + \cdots + p_k^2$$

with the p_i's $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$ (it can for $n = 1$).

But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, $p \geq 0$, **can indeed** be expressed as such a sum of squares, with the p_i's $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$, in fact, with $k = 2^n$.

This solvability over the **scalar** rational functions immediately leads to solvability in the **matrix** case.

Write the factor $X = DN^{-1}$, with $D, N \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, and N diagonal. Define $\ell' = N(\frac{d}{dx})\ell$. Note that $N(\frac{d}{dx})$ is surjective. Then

$$\int_{\mathbb{R}^n} Q_{\Phi'}(N(\frac{d}{dx})\ell)dx = \int_{\mathbb{R}^n} |D(\frac{d}{dx})\ell|^2 dx$$

This implies $N^\top(-\xi)\Phi'(-\xi,\xi)N(\xi) - D^\top(-\xi)D(\xi) = 0.$

Define $\Psi \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ as a (non-unique) solution of

$$(\zeta+\eta)\cdot\Psi'(\zeta,\eta) = N^\top(\zeta)\Phi'(\zeta,\eta)N(\eta) - D^\top(\zeta)D(\eta).$$

This yields the local law

$$\nabla\cdot Q_\Psi(\ell) = Q_\Phi(w) - |D(\tfrac{d}{dt})\ell|^2$$

with $w = M'(\frac{d}{dx})N(\frac{d}{dx})\ell.$

Hence

$$\boxed{\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w),}$$

with $w = M'(\frac{d}{dx})N(\frac{d}{dx})\ell$, an image representation of $\mathfrak{B}$.

$\square$

Since the image representation $w = M'(\frac{d}{dx})N(\frac{d}{dx})\ell$ need not be observable, it may be impossible to 'replace' ℓ by w in the case $\mathtt{n} \geq 1$.

However, in the case $\mathtt{n} = 1$, we can eliminate ℓ, since the image representation $w = M'(\frac{d}{dt})\ell'$ can be taken to be observable, and N can be taken to be I.

Notes:

1. **Local versions require hidden (unobservable) latent variables!**

2. **Ψ, (or (S, F)) is not unique, not even in the conservative case.**

3. **Is Q_Ψ a state function? What is state for PDE's?**

1. and 2. are illustrated already by Maxwell's equations.

RECAP

- **Dissipative systems:** a central notion in the theory of open systems
- **Important problem:** the construction of storage functions
- For linear differential systems (systems described by PDE's) and quadratic differential forms, the equivalence of global and local dissipativeness have been demonstrated
- The theory of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, of linear differential systems, brings system theory squarely into mainstream mathematics via things as the fundamental principle, module theory, computer algebra, Serre's conjecture, factorization questions, etc.
- Issues as the existence of a potential function, conservation laws, dissipativity, etc., are of much interest in physics as well as in systems theory ($\mathcal{H}_2$ and $\mathcal{H}_\infty$ control and filtering).

In this talk, I freely used ideas and results by Oberst, Shankar, Pillai, Rocha, e.a.

More info? Surf to

`http://www.math.rug.nl/~willems/`

Thank you!